DETERMINATION OF THE REACTION ON A CONTOUR IN A CONSTANT-VORTEX FLOW

A. G. Yarmitskii

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The forces acting on a contour in a plane flow of incompressible ideal fluid with a constant vortex have been analyzed by a number of authors [1–3]. Mikuta and Novikov studied a circular constant-vortex flow, while the authors of [2–4] investigated a uniform flow with a transverse velocity gradient. These studies are chiefly concerned with flow around bodies in a rotary hydraulic channel and with wind effects under natural conditions.

Below it is shown that the method of determining the stream function of a disturbed circular flow, developed in [1], can be extended to any constant-vortex flow, in particular, to a uniform flow with a transverse velocity gradient. A formula is obtained for determining the hydrodynamic reaction of any constant-vortex flow on a contour. A finite analytic expression is derived for the aerodynamic force acting on a circular cylinder in a constant-vortex divergent flow.

We assume on the basis of experimental evidence that a contour introduced into a flow with constant vortex Ω does not affect the distribution of the vortex in the flow. The flow over the contour is described by the equation for the stream function ψ:

$$4\frac{\partial^2\Psi}{\partial z\partial\bar{z}} = -\Omega \tag{1}$$

with the condition on the contour

$$\Psi|_L = \text{const}. \tag{2}$$

At a sufficiently great distance from the contour the stream function will be virtually the same as the stream function of the undisturbed flow Ψ_∞, on which the circulation is superimposed, since the disturbances due to the contour progressively decay. Hence it follows that at infinity the stream function Ψ is on the order of

$$\Psi = \Psi_\infty + O(\ln|z|), \tag{3}$$

where $O(\ln|z|)$ stands for expressions increasing as $|z|\to\infty$ not more rapidly than $\ln|z|$.

The boundary value problem (1)–(3) differs from that considered in [1] only with respect to the more general condition at infinity (3).

As shown in [2], the most general constant-vortex flow can be obtained by superimposing a circular constant-vortex flow with center of rotation at the coordinate origin and a potential flow,* so that

$$\Psi = -{}^1/_4\,\Omega\, z\,\bar{z} + \Psi^\circ. \tag{4}$$

Here, Ψ° is the stream function of some potential flow.

In accordance with (1)–(4), the potential flow defined by the function Ψ° is described by the Laplace equation with the following boundary condition and condition at infinity:

$$\Psi^\circ = {}^1/_4\Omega z\bar{z} + \text{const} \quad \text{on } L,$$

$$\Psi^\circ = \Psi^\circ_\infty + O(\ln|z|). \tag{5}$$

Here, Ψ°_∞ is the stream function of the undisturbed potential flow.

Thus, boundary value problem (1)–(3) reduces to the determination of the harmonic function Ψ° satisfying conditions (5).

Let us solve this problem by the method proposed in [1]. We represent the function Ψ° in the form of a sum of harmonic functions $\Psi^{(1)}$ and $\Psi^{(2)}$, on which we impose the following conditions at infinity:

$$\Psi^{(1)} = {}^1/_4\Omega z\bar{z} + \text{const} \quad \text{on} \quad L, \quad \Psi^{(1)} = O(|z|^{-1}), \tag{6}$$

$$\Psi^{(2)} = 0 \text{ on } L, \qquad \Psi^{(2)} = \Psi_\infty^\circ + O(\ln|z|). \tag{7}$$

Then, as it is easy to see, the function

$$\Psi = -{}^1/_4\Omega z\bar{z} + \Psi^{(1)} + \Psi^{(2)} \tag{8}$$

will be the solution of problem (1)–(3).

In order to determine the function $\Psi^{(2)}$ it is possible to use the circle theorem and conformal mapping. The function $\Psi^{(1)}$ for the case of flow over a Zhukovskii profile was found in [1]. For flow over an ellipse with semiaxes a and b, center at the coordinate origin and semiaxis a directed along the x-axis, this function has the form:

$$\Psi^{(1)} = \operatorname{Im} w^{(1)} = \operatorname{Im}\left(\frac{i}{8}\,\Omega r^2\frac{c^2}{\zeta^2}\right)$$

$$\left(\zeta = \frac{z+\sqrt{z^2-c^2}}{2}, \quad c^2 = a^2 - b^2, \quad r = \frac{a+b}{2}\right). \tag{9}$$

Introducing the complex velocity $\bar{v}$, on the basis of (4) we obtain

$$\bar{v} = \bar{v}^* + \bar{v}^\circ \quad (\bar{v}^* = -{}^1/_2 i\Omega\bar{z}). \tag{10}$$

Here, $\bar{v}^*$ is the complex velocity of the circular constant-vortex flow with center of rotation at the coordinate origin, and $\bar{v}^\circ$ is the complex velocity of the potential flow defined by functions $\Psi^{(1)}$ and $\Psi^{(2)}$.

If the flow over the profile is the result of the superposition of two plane steady flows, then in this case the aerodynamic force can be regarded as the sum of the aerodynamic forces corresponding to each of the superimposed flows and the aerodynamic force due to their mutual interference $R^{*\circ}$ (i.e., depending on the velocity components of both flows), so that

$$\bar{R} = \bar{R}^* + \bar{R}^\circ + \bar{R}^{*\circ}. \tag{11}$$

We determine the aerodynamic force due to the circular constant-vortex flow with center of rotation at the coordinate origin and the aerodynamic force due to the mutual interference of the superimposed flows.

In accordance with the Chaplygin-Blasius theorem, using (10), we have

$$\bar{R}^* = \frac{i\rho}{2}\oint_{(L)} \bar{v}^{*2}\,dz = -\frac{i\rho}{8}\,\Omega^2\oint_{(L)}\bar{z}^2\,dz.$$

Using the complex form of the Stokes theorem [1, 2], we obtain

$$\oint_{(L)}\bar{z}^2\,dz = 4i\iint_{(S)}\bar{z}\,dS = 4iS\bar{z}_c \quad (z_c = x_c + iy_c).$$

Here, S is the area bounded by the contour L, and z_c is the position of the center of gravity of that area. Consequently,

$$\bar{R}^* = {}^1/_2\rho\Omega^2 S\bar{z}_c. \tag{12}$$

We now find the aerodynamic force due to the interference of the superimposed flows:

$$\bar{R}^{*\circ} = i\rho\oint_{(L)}\bar{v}^*\bar{v}^\circ\,dz = \frac{1}{2}\rho\Omega\oint_{L}\bar{z}\,dw\,.$$

Here, w is the complex potential of the irrotational flow. As shown in [2],

$$\oint_{(L)}\bar{z}\,dw = \oint_{(L)} z\,d\bar{w} - 2i\oint_{(L)}\Psi^\circ\,d\bar{z}.$$

*The conclusion has been reformulated in my terminology.

Again using the complex form of the Stokes theorem and the first of expressions (5) for Ψ°, we obtain

$$\oint_{(L)} \Psi^\circ d\bar{z} = -\frac{1}{2} i\Omega \iint_{(S)} \bar{z}\, dS = -\frac{1}{2} i\Omega S \bar{z}_c .$$

Hence

$$\oint_{(L)} \bar{z}\, dw = \oint_{(L)} \bar{z}\, d\bar{w} - \Omega S \bar{z}_c .$$

Thus,

$$\bar{R}^{*\circ} = {}^1/_2\, \rho\Omega \left(\oint_{(L)} \bar{z}\, d\bar{w} - \Omega S \bar{z}_c \right). \tag{13}$$

Denoting the component of the aerodynamic force determined by the vorticity of the flow by R^{**}, on the basis of expressions (11-13) we find

$$\bar{R} = \bar{R}^\circ + \bar{R}^{**}.$$

$$\bar{R}^\circ = \frac{i\rho}{2} \oint_{(L)} \bar{v}^{\circ 2}\, dz, \qquad R^{**} = R^* + R^{*\circ} = \frac{1}{2} \rho\Omega \oint_{(L)} z\bar{v}^\circ\, dz. \tag{14}$$

Assuming the absence of singularities in the flow, we will treat the complex velocity $\bar{v}^\circ$ as a holomorphic function of z in the exterior (with respect to the contour L) part of the z-plane. Then, in the neighborhood of an infinitely remote point we have the Laurent series

$$\bar{v}^\circ = \sum_{n=0}^{m} a_n z^n + \sum_{n=1}^{\infty} \frac{a_{-n}}{z^n}. \tag{15}$$

After integrating over a circle of sufficiently large radius, in order to make expansion (15) correct, we express the components of the aerodynamic force R° and R^{**} in terms of the coefficients of that expansion

$$\bar{R}^\circ = -2\pi\rho \sum_{n=0}^{m} a_n a_{-(n+1)}, \qquad R^{**} = \pi\rho\Omega i a_{-2}. \tag{16}$$

Thus, the principal vector of the fluid pressure forces acting on the profile is expressed as follows:

$$R = X + iY = -\pi\rho \left(2 \sum_{n=0}^{m} \bar{a}_n \bar{a}_{-(n+1)} - i a_{-2} \Omega \right). \tag{17}$$

Setting

$$\Omega = 0, \quad a_{-1} = -i\chi,$$

$$a_n = \bar{v}_\infty \ (n = 0), \quad a_n = 0 \ (n \neq 0),$$

where $\chi = \Gamma/2\pi$ is the intensity of the circulation Γ around the profile and $\bar{v}_\infty$ is the complex velocity of the plane-parallel flow at infinity, we obtain the Zhukovskii theorem. Let us consider certain examples.

Ellipse in a uniform flow with transverse velocity gradient. Let the free-stream velocity distribution in the plane $\zeta(\xi, \eta)$ have the form:

$$U = -\Omega\eta + U_\infty, \qquad V = 0.$$

Here, U_∞ is the velocity of the uniform flow at infinity ($\xi \to \infty, \eta = 0$). Then the complex free-stream velocity and its potential part will be, respectively,

$$\bar{v}_\infty = {}^1/_2\, i\Omega(\zeta - \bar{\zeta}) + U_\infty, \qquad \bar{v}_\infty^\circ = {}^1/_2\, i\Omega\zeta + U_\infty.$$

In accordance with the circle theorem [2] the complex velocity of the disturbed potential flow around a circular cylinder of radius r is

$$\bar{v}_\zeta^{(2)} = \frac{1}{2} i\Omega\zeta + U_\infty + \frac{r^2}{\zeta^2} \left(\frac{1}{2} i\Omega \frac{r^2}{\zeta} - U_\infty \right).$$

In the more general case of flow over a cylinder when the free-stream velocity is directed at an angle α to the x-axis and a circulation $\Gamma = 2\pi\chi$ around the cylinder is superimposed on the flow, the complex velocity of the disturbed irrotational flow has the form

$$\bar{v}_\zeta^{(2)} = \frac{1}{2} i\Omega e^{-2i\alpha}\zeta + U_\infty e^{-i\alpha} - \frac{i\chi}{\zeta} - \frac{U_\infty r^2 e^{i\alpha}}{\zeta^2} + \frac{{}^1/_2\, i\Omega r^4 e^{2i\alpha}}{\zeta^3}.$$

Using conformal mapping, we find that the complex velocity of the flow around an elliptic cylinder $\bar{v}_z^{(2)}$ is related with the complex velocity of flow around a circular cylinder $\bar{v}_\zeta^{(2)}$ by the expression

$$\bar{v}_z^{(2)} = \frac{\zeta^2}{\sqrt{z^2 - c^2}} \bar{v}_\zeta^{(2)}.$$

Consequently, the complex velocity for an elliptic cylinder associated with the stream function $\Psi^{(2)}$ has the form

$$\bar{v}^{(2)}{}_z = \frac{1}{2} i\Omega e^{-2i\alpha} \frac{\zeta^2}{\sqrt{z^2 - c^2}} + U_\infty e^{-i\alpha} \frac{\zeta}{\sqrt{z^2 - c^2}} - i\chi \frac{1}{\sqrt{z^2 - c^2}} - U_\infty r^2 e^{i\alpha} \frac{1}{\zeta\sqrt{z^2 - c^2}} + \frac{1}{2} i\Omega r^4 e^{2i\alpha} \frac{1}{\zeta^2 \sqrt{z^2 - c^2}}.$$

We now find the complex velocity $\bar{v}_z^{(1)}$ corresponding to the function $\Psi^{(1)}$:

$$\bar{v}_z{}^{(1)} = \frac{d\,[w^{(1)}(\varphi(z))]}{dz},$$

where, in accordance with the foregoing,

$$w^{(1)}(\zeta) = -\frac{1}{8} i\Omega r^2 \frac{c^2}{\zeta^2}, \qquad \zeta = \varphi(z) = \frac{1}{2}(z + \sqrt{z^2 - c^2}).$$

Consequently, for $\bar{v}_z^{(1)}$ we obtain

$$\bar{v}_z{}^{(1)} = -\frac{1}{4} i\Omega r^2 \frac{c^2}{\zeta^2 \sqrt{z^2 - c^2}}. \tag{18}$$

The complex velocity for an elliptic cylinder in the potential part of the disturbed flow

$$\bar{v}^\circ = \bar{v}_z^{(1)} + \bar{v}_z^{(2)}. \tag{19}$$

In the neighborhood of an infinitely remote point

$$\bar{v}^\circ = a_1 z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots,$$

where

$$a_1 = {}^1/_2 i\Omega e^{-2i\alpha}, \quad a_0 = U_\infty e^{-i\alpha},$$

$$a_{-1} = -i\chi, \quad a_{-2} = ({}^1/_4 c^2 e^{-i\alpha} - r^2 e^{i\alpha})\, U_\infty.$$

Starting from (17), in this case we write

$$R = -\pi\rho[2(\bar{a}_0\bar{a}_{-1} + \bar{a}_1\bar{a}_{-2}) - ia_{-2}\Omega] =$$

$$= -i[\rho U_\infty \Gamma + \pi\rho\Omega U_\infty (a + b)\,(a\sin^2\alpha + b\cos^2\alpha)] b^{i\alpha}.$$

Hence the lift force is

$$F = -\rho U_\infty \Gamma - \pi\rho\Omega U_\infty (a + b)\ (a\sin^2\alpha + b\cos^2\alpha). \tag{20}$$

If there is no circulation Γ around the contour, the result obtained coincides with the result of [3].* In this case setting $a = b = r$, we find

$$F = -2\rho U_\infty \Gamma', \quad \Gamma' = \Omega\pi r^2. \tag{21}$$

Expression (21) coincides with the known expression for a circular cylinder [3, 4]. If in (20) we set $\Gamma = 0$, $b = 0$, $0 \le \alpha \le \pi/2$, we obtain the lift force for noncurculatory flow over a plate

$$F = -\rho U_\infty \Omega\pi a^2 \sin^2\alpha. \tag{22}$$

This force reaches a maximum at $\alpha = \pi/2$, when the plate is arranged perpendicular to the flow. This is also in agreement with the conclusions of [3].

Ellipse in a circular constant-vortex flow. Let the center of rotation be located at the point $z_0 = x_0 + iy_0$. Then the velocity components at any point of the undisturbed flow can be represented in the

* The difference in signs is attributable to the reversal of the sign rule for the vortex Ω.

form $U = -\omega(y - y_0)$, $V = \omega(x - x_0)$, where $\omega = \Omega/2$ is the angular velocity of the flow.

Hence the complex velocity of the undisturbed flow is

$$\bar{v}_\infty = -i\omega(\bar{z} - \bar{z}_0).$$

Introducing relation (10), we isolate the complex velocity of the potential part of this flow,

$$\bar{v}_\infty^\circ = i\omega\bar{z}_0.$$

The case of an elliptic cylinder in a potential flow with constant velocity at infinity has been thoroughly investigated [5]; the complex velocity has the form

$$\bar{v}_z^{(2)} = \frac{1}{2}\left[\bar{v}_\infty^\circ\left(1 + \frac{z}{\sqrt{z^2 - c^2}}\right) + \frac{(a+b)^2}{c^2} v_\infty^\circ\left(1 - \frac{z}{\sqrt{z^2 - c^2}}\right)\right] - \frac{i\chi}{\sqrt{z^2 - c^2}}.$$

Using (18) and (19), we find

$$\bar{v}^\circ = \bar{v}_z^{(2)} - i\Omega r^2 \frac{c^2}{(z + \sqrt{z^2 - c^2})^2\sqrt{z^2 - c^2}}. \tag{23}$$

A Laurent expansion of the right-hand side of the latter expression in the neighborhood of an infinitely remote point yields

$$\bar{v}^\circ = a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \ldots,$$

$$a_0 = \bar{v}_\infty^\circ, \quad a_{-1} = -i\chi, \quad a_{-2} = {}^1/_4\,[\bar{v}_\infty^\circ c^2 - v_\infty^\circ(a+b)^2].$$

We now apply Eq. (17),

$$R = -2\pi\rho(\bar{a}_0\bar{a}_{-1} - i\omega a_{-2}) = -i\rho v_c(\Gamma + \Gamma'), \tag{24}$$

where $v_c = -i\omega z_0$ is the velocity of the undisturbed flow on the cylinder axis

$$\Gamma' = \frac{1}{2}\pi\omega\,[(a+b)z_0 + (a-b)\bar{z}_0]\,\frac{a+b}{z_0}.$$

Setting $z_0 = iy_0$, $a = b = r$, we find

$$\bar{R} = i\rho v_c(\Gamma + \Gamma') \quad (v_c = \omega y_0, \ \Gamma' = \Omega\pi r^2). \tag{25}$$

This result coincides with the known expression giving the aerodynamic force for a circle [1], when the center of rotation of the flow is located on the imaginary axis.

If we set $b = 0$, $a = c$ in (24), we obtain an expression for the aerodynamic force acting on a flat plate of length 2c

$$R = -i\rho v_c(\Gamma + \Gamma'), \quad \Gamma' = {}^1/_2\omega\pi c^2(1 + \bar{z}_0/z_0), \tag{26}$$

where v_c is the velocity of the undisturbed flow at the center of the plate.

We determine the circulation Γ from the condition of finite velocity at the trailing edge of the plate. For this purpose we analyze the expression for the complex velocity. On the basis of (10) and (23) we find

$$\bar{v} = -i\omega\bar{z} + U^\circ - \frac{i}{\sqrt{z^2 - c^2}}\left(V^\circ z + \chi + \frac{1}{2}\omega\frac{c^4}{(z + \sqrt{z^2 - c^2})^2}\right).$$

Here, U° and V° are the velocity components of the potential part of the undisturbed flow. At an arbitrary value of the circulation $\Gamma = 2\pi\chi$ and $z = \pm c$ the velocity has infinite values, which corresponds to flow over sharp leading and trailing edges. We now impose on Γ the condition of finite velocity at the trailing edge ($z = c$), as required by the Zhukovskii-Chaplygin postulate. We then obtain

$$\Gamma = 2\pi\chi = -2\pi c(V^\circ + {}^1/_2\omega c) = \pi\omega c(z_0 + \bar{z}_0 - c).$$

Substituting the values of Γ and Γ' into the first of expressions (26), we find

$$R = -\pi\rho\omega^2 c[(\bar{z}_0 + \bar{z}_0)\,({}^1/_2 c + z_0) - cz_0].$$

Starting from this, we can write

$$F_\alpha = {}^1/_2\,\pi\rho\omega^2 c^2\,|z_0|\sin 2\alpha,$$

$$F_{\alpha + {}^1/_2\pi} = 2\pi\rho\omega^2 c\,|z_0|\,({}^1/_2 c\cos^2\alpha + |z_0|\sin\alpha). \tag{27}$$

Here, α is the local angle of attack at the center of the plate, F_α is the component of the aerodynamic force in the direction of the free-stream velocity at the center of the plate, and $F_{\alpha + {}^1/_2\pi}$ the component of the aerodynamic force in the direction $\alpha + \pi/2$.

As distinct from a potential flow over a flat plate, where the angle of attack can always be selected so that the lift force vanishes, in this case the aerodynamic force is always nonzero. In fact, it is not possible to find an angle such that $F_\alpha = 0$ and $F_{\alpha+\pi/2} = 0$ simultaneously.

This conclusion is fully consistent with that reached in [1].

In analyzing expressions (20)–(22) and (24)–(26), we note that, as distinct from a potential flow, in a constant-vortex flow an aerodynamic force is observed even in the absence of circulation Γ. This force is proportional to the modulus of the vortex vector.

Circular cylinder in a constant-vortex divergent flow. Suppose that at the center of rotation of a circular constant-vortex flow $z = z_0$ there is a source of strength m located outside a cylinder L of radius r whose axis passes through the coordinate origin C. The complex velocity of the undisturbed flow and the potential part of that flow will then take the following form:

$$\bar{v}_\infty = \frac{m}{z - z_0} - i\omega(\bar{z} - \bar{z}_0), \quad \bar{v}_\infty^\circ = \frac{m}{z - z_0} + i\omega\bar{z}_0.$$

Using the circle theorem [2], we can write the complex velocity of the corresponding disturbed flow in the form:

$$\bar{v}^\circ = \frac{m}{z - z_0} + \frac{m}{z - r^2/\bar{z}_0} - \frac{m + i\chi}{z} + i\omega\bar{z}_0 + \frac{r^2}{z^2}i\omega z_0.$$

In this case, as it is easy to see, $v_z^{(1)} \equiv 0$.

Since there is a singularity in the flow, it is not possible to apply Eq. (17) directly. Therefore, in order to calculate the aerodynamic force acting on the cylinder we employ expression (14).

Locating the source on the real axis at the point $z = z_0$, from the residue theorem we obtain

$$\bar{R} = 2\pi\rho r\left(\frac{k}{k^2 - 1}U_c^2 - \frac{1}{k}V_c^2\right) - i\rho\bar{v}_c\Gamma,$$

$$\bar{v}_c = U_c - iV_c, \quad k = |z_0|/r. \tag{28}$$

Here, U_c and V_c are the velocity components of the undisturbed flow on the cylinder axis at the coordinate origin.

Analyzing (28), we note that, as in the previous examples, the aerodynamic force is nonzero even in the absence of circulation Γ around the cylinder. This force pulls the cylinder toward the source, with $U_c > V_c$. If the force is directed away from the source, $U_c < V_c$. As $k \to \infty$, i.e., as we approach the conditions of a cylinder in a homogeneous flow, this force vanishes. In the latter case, in the presence of a circulation Γ around the cylinder, formula (28) yields Zhukovskii's theorem, since as k tends to infinity we obtain $R = i\rho v_c\Gamma$.

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